**Inequality 1**

1. Prove the inequalty $a^{2}+b^{2}\geq 2ab$. If $x+y+z=c$ , show that $x^{2}+y^{2}+z^{2}\geq \frac{1}{3}c^{2}.$

 $\left(a-b\right)^{2}\geq 0⟺a^{2}+b^{2}\geq 2ab$

 $3\left(x^{2}+y^{2}+z^{2}\right)=x^{2}+y^{2}+z^{2}+\left(x^{2}+y^{2}\right)+\left(y^{2}+z^{2}\right)+\left(z^{2}+x^{2}\right)$

 $\geq x^{2}+y^{2}+z^{2}+2xy+2yz+2zx=\left(x+y+z\right)^{2}=c^{2}$

 $∴x^{2}+y^{2}+z^{2}\geq \frac{1}{3}c^{2}$

2. Find the solution set for the inequlaity: $\left|\frac{4}{x-1}\right|\geq 3-\frac{3}{x}$

 Note that $x\ne 0,1$.

 Case 1 If $x < 1$,

 (a) If $x > 0$, that is $0 < x < 1$

 $-\frac{4}{x-1}\geq 3\left(\frac{x-1}{x}\right)⟹\frac{4}{1-x}\geq 3\left(\frac{x-1}{x}\right)⟹4x\geq -3\left(1-x\right)^{2}⟹3x^{2}-2x+3\geq 0$

 Since $∆=\left(-2\right)^{2}-4\left(3\right)\left(3\right)<0$, $3x^{2}-2x+3\geq 0$ is always true.

 $0 < x< 1$ is a solution.

 (b) If $x< 0$,

 $-\frac{4}{x-1}\geq 3\left(\frac{x-1}{x}\right)⟹\frac{4}{1-x}\geq 3\left(\frac{x-1}{x}\right)⟹4x\leq -3\left(1-x\right)^{2}⟹3x^{2}-2x+3\geq 0$

 Since $∆=\left(-2\right)^{2}-4\left(3\right)\left(3\right)<0$, $3x^{2}-2x+3\leq 0$ is always false.

 There is no solution in this case.

 Case 2 If $x > 1$,

 $\frac{4}{x-1}\geq 3\left(\frac{x-1}{x}\right)⟹4x\geq 3\left(x-1\right)^{2}⟹3x^{2}-10x+3\leq 0⟹\left(3x-1\right)\left(x-3\right)\leq 0$

 $⟹\frac{1}{3}\leq x\leq 3$

 Together with $x> 1$, the solution in this case is $1<x\leq 3$.

 Combining case 1 and 2, the solution is $0 < x < 1$ and $1<x\leq 3$.

3. Solve $\left|5-2x\right|\leq 3x+10$ , $\left|5-2x\right|<3x+10$

 **Method 1**

 Since $\left|5-2x\right|\geq 0$, in order the given inequality to have solution,$3x+10\geq 0⟹ x\geq -\frac{10}{3}$

 If $x\geq -\frac{10}{3}$, $\left|5-2x\right|^{2}\leq \left(3x+10\right)^{2}⟹25-20x+4x^{2}\leq 9x^{2}+60x+100$

 $⟹5x^{2}+80x+75\geq 0⟹x^{2}+16x+15\geq 0⟹\left(x+1\right)\left(x+15\right)\geq 0$

 $⟹x\leq -15 or x\geq -1$

 Since $x\geq -\frac{10}{3}$, the solution is $x\geq -1$.

 **Method 2**

$\left|5-2x\right|\leq 3x+10$

 $\left\{\begin{array}{c}5-2x\geq 0\\5-2x\leq 3x+10\end{array}\right.$ or $\left\{\begin{array}{c}5-2x\leq 0\\-\left(5-2x\right)\leq 3x+10\end{array}\right.$

 $\left\{\begin{array}{c}x\leq \frac{5}{2}\\x\geq -1\end{array}\right.$ or $\left\{\begin{array}{c}x\geq \frac{5}{2}\\x\geq -15\end{array}\right.$

 $-1\leq x\leq \frac{5}{2}$ or $x\geq \frac{5}{2}$

 $∴x\geq -1$.

**4.** Sketch on the same axes, the graphs of $y=\left|2x+1\right|$ and $y=1-x^{2}$.

 Hence, solve the inequality $\left|2x+1\right|\geq 1-x^{2}$ .

 Solve :

 $\left\{\begin{array}{c}y=-\left(2x+1\right)\\y=1-x^{2}\end{array}\right.$

 $\left(x=\sqrt{3}+1, y=-2 \sqrt{3}-3\right) $

$$or \left(x=1-\sqrt{3}, y=2 \sqrt{3}-3\right)$$

 Solution for

$$\left|2x+1\right|\geq 1-x^{2}$$

 is $x\leq 1-\sqrt{3} or x\geq 0$

**5.** Show that

 $-2\leq \frac{4x}{4x^{2}+2x+1}\leq \frac{2}{3}$ , where x is real.

 **Method 1 (Quadratics)**

 Let $y=\frac{4x}{4x^{2}+2x+1}$

 $y\left(4x^{2}+2x+1\right)=4x$

 $\left(4y\right)x^{2}+\left(2y-4\right)x+y=0$

 Since x is real, $∆=\left(2y-4\right)^{2}-4\left(4y\right)y\geq 0$

 $16-16 y-12 y^{2}\geq 0$

 $3y^{2}+4y-4\leq 0$

 $\left(y+2\right)\left(3y-2\right)\leq 0$

 $-2\leq y\leq \frac{2}{3}$

 Maximum is $\frac{3}{2}$ . Minimum is $-2$.

**Method 2 (Calculus)**

 Let $y=\frac{4x}{4x^{2}+2x+1}$

 $\frac{dy}{dx}=\frac{\left(4x^{2}+2x+1\right)\left(4\right)-4x\left(8x+2\right)}{\left(4x^{2}+2x+1\right)^{2}}=\frac{-4\left(4x^{2}-1\right)}{\left(4x^{2}+2x+1\right)^{2}}$

For critical values, $\frac{dy}{dx}=0⟹4x^{2}-1=0⟹x=\pm \frac{1}{2}$

 For $x<-\frac{1}{2}, \frac{dy}{dx}<0$

 For $-\frac{1}{2}<x<\frac{1}{2}, \frac{dy}{dx}>0$

For $x>\frac{1}{2}, \frac{dy}{dx}<0$

 When $x=-\frac{1}{2}$, y is a minimum, $y=\frac{4\left(-\frac{1}{2}\right)}{4\left(-\frac{1}{2}\right)^{2}+2\left(-\frac{1}{2}\right)+1}=-2$

 When $x=\frac{1}{2}$, y is a maximum, $y=\frac{4\left(\frac{1}{2}\right)}{4\left(\frac{1}{2}\right)^{2}+2\left(\frac{1}{2}\right)+1}=-2$

Therefore, $-2\leq y\leq \frac{2}{3}$

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